REDUCTION OF THE PROBLEM OF MOTION OF A HEAVY RIGID BODY WITH ONE FIXED POINT TO A SINGLE EQUATION. NEW PARTICULAR SOLUTION OF THE ABOVE PROBLEM

(SVEDENIE ZADACHI O DVIZHENII TIAZHELOGO TVERDOGO TELA, IMEIUSHCHEGO NEPODVIZHNUIU TOCHKU, K ODNOMU URAVNENIIU. NOVOE CHASTNOE RESHENIE ETOI ZADACHI)

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Up to the present time, thirteen particular solutions of this problem were reported by various authors. These solutions can be devided into two groups. First group will contain the solutions found under the condition that the center of gravity lies on the principal axis of the ellipsoid of gyration. We find there solutions by Zhukovskii [1] (*), Lagrange, Kowalewski [3], Chaplygin [4], three solutions utilizing polynomial integrals [5] and the solution of Sretenskii [6] generalizing the Goriachev-Chaplygin case of integrability [7]. In the remaining five solutions, conditions defining the position of the center of gravity are less restrictive. It can be arbitrary when the rotation of the body is uniform (solution with three linear integrals [8]). In the solution with two linear integrals [9] and in three solutions with one linear integral [10 to 12], the center of gravity lies on the principal plane. Last five equations have a common feature. In each of them a linear integral occurs.

The solution presented in this paper is outside the first group since the center of gravity lies on the principal plane (not on the principal axis). Integrals in it are however, unlike in the second group, nonlinear.

The problem of motion of a heavy rigid body with one fixed point, is reduced to a system of two differential equations of first order [13]. This system is equivalent to one differential equation of second order, which is, in general, very complex. If however one of the special coordinate axes coincides with the principal axis, then the problem can be reduced to one, relatively simple equation. This method is utilized to obtain another particular solution to our problem.

1. Let λ, λ_1 and λ_2 be a gyrostatic moment constant with respect to the body and $x + \lambda$, $y + \lambda_1$ and $z + \lambda_2$ be the angular moment of the

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*) Zhukovskii gave the integrals and geometrical interpretation of the motion of the body for the case when the center of gravity coincides with the fixed point, and the gyrostatic moment is arbitrary. When the latter becomes zero, Zhukovskii's solution reduces to Euler's solution. Quadratures in Zhukovski's solution were later referred to by Volterra [2].

system relative to the fixed point. Let further γ, γ_1 and γ_2 be an invariant vector in the direction of the force of gravity, the modulus Γ of which is equal to the product of the mass of the system and the distance of the center of gravity from the fixed point. Denoting the components of the gyration tensor in the special coordinate system by a, a_1, a_2, b_1 and b_2, we can write the equations of motion in the form [13]

$$dx / dt = (a_2 z + b_2 x)(y + \lambda_1) - (a_1 y + b_1 x)(z + \lambda_2) \tag{1.1}$$
$$dy / dt = (ax + b_1 y + b_2 z)(z + \lambda_2) - (a_2 z + b_2 x)(x + \lambda) - \gamma_2$$

$$d\gamma / dt = (a_2 z + b_2 x)\gamma_1 - (a_1 y + b_1 x)\gamma_2 \tag{1.2}$$
$$d\gamma_1 / dt = (ax + b_1 y + b_2 z)\gamma_2 - (a_2 z + b_2 x)\gamma$$

Out of six equations, we have written above only four, which shall be used later. We shall replace the remaining two equations with the integrals

$$ax^2 + a_1 y^2 + a_2 z^2 + 2(b_1 y + b_2 z)x - 2\gamma = 2E \tag{1.3}$$
$$(x + \lambda)\gamma + (y + \lambda_1)\gamma_1 + (z + \lambda_2)\gamma_2 = k \tag{1.4}$$

Let one of the special coordinate axes (e.g. the third one) coincide with the principal axis

$$b_2 = 0 \tag{1.5}$$

and let the gyrostatic moment be orthogonal to this axis

$$\lambda_2 = 0 \tag{1.6}$$

Conditions (1.5) and (1.6) represent exactly the restraints imposed on the parameters of the system, in presence of which the problem reduces to one, relatively simple equation.

From (1.1) (in the following the subscript of b_1 will be omitted), we have

$$dy / dt = z[(a - a_2)x + by - a_2\lambda] - \gamma_2 \tag{1.7}$$
$$dx / dt = -zX(y, x), \qquad X(y, x) = (a_1 - a_2)y + bx - a_2\lambda_1$$

elimination of t results in

$$\gamma_2 = z\frac{dY}{dx}, \qquad Y(y(x), x) = \frac{a_1 - a_2}{2}y^2 + (bx - a_2\lambda_1)y + \frac{a - a_2}{2}x^2 - a_2\lambda x + n_* \tag{1.8}$$

where n_* is a constant. Substitution of (1.5) to (1.8) into (1.2) eliminates from the latter the variable z

$$X\frac{d\gamma}{dx} + a_2\gamma_1 = (a_1 y + bx)\frac{dY}{dx}, \qquad X\frac{d\gamma_1}{dx} - a_2\gamma = -(by + ax)\frac{dY}{dx}$$

and gives

$$\gamma + i\gamma_1 = (\gamma^\circ + i\gamma_1^\circ)\exp ia_2\int_{x_0}^{x}\frac{d\sigma}{X(y(\sigma), \sigma)} +$$
$$+\int_{x_0}^{x} F(y(\sigma), \sigma)\frac{dY(y(\sigma), \sigma)}{d\sigma}\left(\exp ia_2\int_{\sigma}^{x}\frac{d\tau}{X(y(\tau), \tau)}\right)d\sigma \tag{1.9}$$

where

$$F(y(x), x) = \frac{(a_1 - ib)y(x) + (b - ia)x}{X(y(x), x)} \tag{1.10}$$

Let us now eliminate z from the integrals (1.3) and (1.4)

$$a_2 z^2 = 2\gamma + 2E - a_1 y^2 - 2bxy - ax^2, \qquad (x + \lambda)\gamma + (y + \lambda_1)\gamma_1 + z^2\frac{dY}{dx} = k \tag{1.11}$$

to obtain

$$\left\{2\frac{dY}{dx} + (x + \lambda)a_2\right\}\gamma + (y + \lambda_1)a_2\gamma_1 = a_2 k + (a_1 y^2 + 2bxy + ax^2 - 2E)\frac{dY}{dx}$$

Real part of the product

$$(\gamma + i\gamma_1)\left\{2\frac{dY}{dx} + (x+\lambda)\, a_2 - i\,(y+\lambda_1)\, a_2\right\}$$

is found on the left-hand side, hence utilizing (1.9), we obtain

$$\mathrm{Re}\left\{\left[2\frac{dY}{dx} + (x+\lambda)\, a_2 - i\,(y+\lambda_1)\, a_2\right]\left[(\gamma^\circ + i\gamma_1^\circ)\exp ia_2\int\limits_{x_0}^{x}\frac{d\sigma}{X\,(y\,(\sigma),\,\sigma)} + \right.\right.$$

$$\left.\left. + \int\limits_{x_0}^{x} F\,(y\,(\sigma),\,\sigma)\,\frac{dY\,(y\,(\sigma),\,\sigma)}{d\sigma}\left(\exp ia_2\int\limits_{\sigma}^{x}\frac{d\tau}{X\,(y\,(\tau),\,\tau)}\right) d\sigma\right\} =$$

$$= a_2 k + (a_1 y^2 + 2bxy + ax^2 - 2E)\frac{dY}{dx} \qquad (1.12)$$

Substituting into the latter X, Y and F from (1.7), (1.8) and (1.10), we obtain integro-differential equation defining the function $y = y(x)$. With the latter known, the relation between γ and γ_1, and x, can be found from (1.9). Next $z = z(x)$ is determined by (1.11), afterwhich (1.8) gives us $\gamma_2 = \gamma_2(x)$ and (1.7) produces the relationship between x and t after completing the quadrature.

2. Analogous results can be obtained by another method mentioned in the introduction. Two equations which under the conditions (1.5) and (1.6) have only even powers of z, are

$$(y+\lambda_1)\, X\frac{dz^2}{dx} - \left[(x+\lambda)\, a_2 + 2\frac{dY}{dx}\right] z^2 = \Phi\,(y,\, x)$$

$$\left\{\frac{1}{2}X\frac{dz^2}{dx} + (a_1 y + bx)(x+\lambda) - (ax+by)(y+\lambda_1)\right\}^2 + \qquad (2.1)$$

$$+\left(\frac{dY}{dx}\right)^2 z^2 + \left\{\frac{1}{2}(ax^2 + a_1 y^2 + a_2 z^2) + bxy - E\right\}^2 = \Gamma^2$$

$$\Phi\,(y,\, x) = 2\,(ax+by)(y+\lambda_1)^2 + (x+\lambda)(ax^2 - a_1 y^2 - 2a_1\lambda_1 y - 2b\lambda_1 x - 2E) - 2k$$

First of them is linear in z^2. Substituting

$$z^2 = z_0^2\exp\int\limits_{x_0}^{x}\frac{(\sigma+\lambda)\, a_2 + 2dY\,(y\,(\sigma),\,\sigma)\,/\,d\sigma}{(y\,(\sigma)+\lambda_1)\, X\,(y\,(\sigma),\,\sigma)}\, d\sigma +$$

$$+\int\limits_{x_0}^{x}\frac{\Phi\,(y\,(\sigma),\,\sigma)}{(y\,(\sigma)+\lambda_1)\cdot X\,(y(\sigma),\,\sigma)}\left(\exp\int\limits_{\sigma}^{x}\frac{(\tau+\lambda)\, a_2 + 2dY\,(y\,(\tau),\,\tau)\,/\,d\tau}{(y\,(\tau)+\lambda_1)\, X\,(y\,(\tau),\,\tau)}\, d\tau\right) d\sigma$$

into (2.1), we obtain the equation connecting y and x, equivalent to (1.12). In the following however, (1.12) will be found more conveneient in use.

3. If $a_2 = a_1$, Equation (1.12) can be radically simplified. This condition, together with (1.5), means that the first coordinate axis on which we have the center of gravity, is perpendicular to the circular cross section of the gyratory ellipsoid [14].

We shall now introduce a new variable ξ

$$x = \xi + \varkappa\lambda_1 \qquad (\varkappa = a_1\,/\,b) \qquad (3.1)$$

Relationships from Section 1, will now take the form (3.2)

$$X = b\xi,\; y\,(Y\,(\xi),\,\xi) = \frac{Y\,(\xi) - n}{b\xi} - \frac{a - a_1}{2b}(\xi + 2\varkappa\lambda_1) + \varkappa\lambda \quad \left(n = n_* + \frac{a - a_1}{2}\varkappa^2 - a_1\lambda\lambda_1\right)$$

$$\gamma + i\gamma_1 = (\gamma^\circ + i\gamma_1^\circ)\left(\frac{\xi}{\xi_0}\right)^{i\varkappa} + \int\limits_{\xi_0}^{\xi} \frac{(a_1 - ib)\, y\,(Y(\tau), \tau) + (b - ia)(\tau + \varkappa\lambda_1)}{b\tau}\left(\frac{\xi}{\tau}\right)^{i\varkappa}\frac{dY}{d\tau}\,d\tau \tag{3.3}$$

$$a_1 z^2 = 2\gamma + 2E - a_1 y^2 - 2\,(b\xi + a_1\lambda_1)\, y - a\,(\xi + \varkappa\lambda_1)^2$$

$$\gamma_2 = z dY \,/\, d\xi, \qquad d\xi \,/\, dt = -\, bz\xi \tag{3.4}$$

Dependence of Y on ξ is found from Equation

$$\operatorname{Re}\left\{\left[2\frac{dY}{d\xi} + a_1(\xi + \varkappa\lambda_1 + \lambda - iy\,(Y,\, \xi) - i\lambda_1)\right]\left[(\gamma^\circ + i\gamma_1^\circ)\left(\frac{\xi}{\xi_0}\right)^{i\varkappa} + \right.\right.$$
$$\left.\left. + \int\limits_{\xi_0}^{\xi}\frac{(a_1 - ib)\, y\,(Y(\tau), \tau) + (b - ia)(\tau + \varkappa\lambda_1)}{b\tau}\left(\frac{\xi}{\tau}\right)^{i\varkappa}\frac{dY}{d\tau}\,d\tau\right]\right\} = \tag{3.5}$$
$$= a_1 k + \{a_1 y^2 + 2\,(b\xi + a_1\lambda_1)\, y + a\,(\xi + \varkappa\lambda_1)^2 - 2E\}\frac{dY}{d\xi}$$

4. Simplest particular solutions of (3.5) can be sought in the class of polynomials. Let us assume for example, that

$$Y(\xi) - n = c\xi^2 + 2c_1\xi + c_0 \tag{4.1}$$

(constants c, c_1 and c_0 which are to be determined, are assumed real). Then, from (3.2) we have

$$by = \left(c - \frac{a - a_1}{2}\right)\xi + 2c_1 + a_1\lambda - (a - a_1)\,\varkappa\lambda_1 + \frac{c_0}{\xi} \tag{4.2}$$

Putting (4.1) and (4.2) into (3.3) and choosing the constant $\gamma^\circ + i\gamma_1^\circ$ so that the resulting expression does not contain ξ in the $i\varkappa$-th power, we obtain (4.3)

$$\gamma = s_0 + s_1\xi + s\xi^2, \qquad \gamma_1 = 2\,\frac{c_0 c_1}{b\xi} + s_0' + s_1'\xi + s'\xi^2 \qquad s_0 = 2\left\{\frac{cc_0 + 2c_1^2}{a_1} + c_1(\lambda + \varkappa\lambda_1)\right\}$$

$$s_1 = \frac{2}{(b^2 + a_1^2)\, b}\left\{a_1\,[2a_1 b\lambda + (b^2 + 2a_1^2 - aa_1)\,\lambda_1]\;\, c + (6a_1 c + b^2 + a_1^2)\, bc_1\right\}$$

$$s = \frac{6a_1 c + 4b^2 + 3a_1^2 - aa_1}{4b^2 + a_1^2}\; c$$

$$s_0' = \frac{2}{b}\left\{cc_0 + 2c_1^2 + \left(a_1\lambda + \frac{b^2 - aa_1 + a_1^2}{b}\,\lambda_1\right)c_1\right\} \tag{4.4}$$

$$s_1' = \frac{2a_1\,[(a_1^2 - b^2)\,\lambda - (a - a_1)\, a_1\varkappa\lambda_1]\; c + [6\,(a_1^2 - b^2)\, c - (a - a_1)(b^2 + a_1^2)]\; c_1}{(b^2 + a_1^2)\, b}$$

$$s' = \frac{2\,(a_1^2 - 2b^2)\, c - (a - a_1)\, a_1^2 - 2ab^2}{(4b^2 + a_1^2)\, b}\; c$$

Now (3.4) becomes

$$b^2 z^2 = -\frac{c_0^2}{\xi^2} + \frac{m_1}{\xi} + m_2 + m_3\xi + m\xi^2 \tag{4.5}$$

$$\gamma_2 = \frac{2}{b}\left(c + \frac{c_1}{\xi}\right)\sqrt{-c_0^2 + m_1\xi + m_2\xi^2 + m_3\xi^3 + m\xi^4} \tag{4.6}$$

$$t = -\int\limits_{\xi_0}^{\xi}\frac{d\xi}{\sqrt{-c_0^2 + m_1\xi + m_2\xi^2 + m_3\xi^3 + m\xi^4}} \tag{4.7}$$

where

$$m_1 = -\,2c_0\,\{2c_1 + a_1\lambda - (a - a_1)\,\varkappa\lambda_1 + b\lambda_1\}$$

$$m_2 = 2\,\frac{2b^2 - a_1^2}{a_1^2}(cc_0 + 2c_1^2) - 4b\lambda_1 c + 4\left(\frac{b^2 - a_1^2}{a_1}\,\lambda + \frac{b^2 + aa_1 - a_1^2}{b}\,\lambda_1\right)c_1 +$$
$$+ (a - a_1)\, c_0 + (b^2 - aa_1 + a_1^2)\,[(a - a_1)\,\varkappa\lambda_1 - 2\lambda a_1] - a_1^2(\lambda^2 + \lambda_1^2) + 2\,\frac{b^2}{a_1}\,E \tag{4.8}$$

$$m_3 = \frac{2c}{(b^2+a_1^2)}\{2(5b^2-a_1^2)c_1+(3b^2-a_1^2)a_1\lambda+[b^3+(2a_1-a)a_1b+(a-a_1)a_1^2\varkappa]\lambda_1\}+$$
$$+2(a-a_1)c_1-(2b^2+a_1^2-aa_1)\lambda-[(3a_1-a)b+(a-a_1)^2\varkappa]\lambda_1 \tag{4.8}$$
$$m = 2\frac{6c+2a_1-a}{4b^2+a_1^2}b^2c-c^2+(a-a_1)c-\frac{1}{4}(a-a_1)^2-b^2$$

Relations (3.1), (4.2), (4.5), (4.3) and (4.6) give the basic variables x, y, z, γ, γ_1 and γ_2 as functions of ξ, which, by (4.7), is an elliptic function of time.

We can complete the solution by showing conditions satisfied by the coefficients c, c_1 and c_2 of the polynomial (4.1). The latter should convert (3.5) into an identity. Putting (4.1) into (3.5), let us utilize (4.2) and (4.3). Comparing the coefficients of like powers of ξ, we obtain six relations. Two of them vanish for any c, c_1 and c_0, while the remaining four can, using the notation of (4.4) and (4.8), be written as

$$b^2s+\left(c-\frac{a-a_1}{2}\right)bs'+2mc=0$$

$$2mc_1+2m_3c+b^2s_1+(a_1\lambda_1+b\lambda)bs+\left(c-\frac{a-a_1}{2}\right)bs'+$$
$$+[2bc_1+a_1b\lambda-(a-a_1)a_1\lambda_1+b^2\lambda_1]s'=0 \tag{4.9}$$

$$2m_3c_1+2m_2c+b^2s_0+(a_1\lambda_1+b\lambda)bs_1+\left(c-\frac{a-a_1}{2}\right)bs_0'+$$
$$+[2bc_1+a_1b\lambda-(a-a_1)a_1\lambda_1+b^2\lambda_1]s_1'+c_0bs'=0 \tag{4.10}$$

$$bk=\frac{2}{b}\left\{m_2c_1+m_1c+c_0c_1\left(c-\frac{a-a_1}{2}\right)\right\}+(a_1\lambda_1+b\lambda)s_0+$$
$$+[2c_1+a_1\lambda+b\lambda_1-(a-a_1)\varkappa\lambda_1]s_0'+c_0s_1' \tag{4.11}$$

Putting (4.4) and (4.8) into (4.9) we find c and c_1

$$6c=2R-a-a_1$$

$$6(4b^2+a_1^2)bc_1=\left\{\frac{b^2+a_1^2}{R}[6b^2+(2a_1-a)a_1]-(7b^2+a_1^2)a_1\right\}b\lambda-$$
$$-\left\{\frac{b^2+a_1^2}{R}[(7a_1-2a)b^2+2a_1(a_1^2-aa_1+a^2)]-2b^4-2(3a_1-a)a_1b^2-(a+a_1)a_1^3\right\}\lambda_1$$
$$R=\pm\sqrt{3b^2+a_1^2-aa_1+a^2}$$

Energy constant E is included in m_2 and can be found from (4.10), while constant k is given by (4.11).

Substitution of (4.3) and (4.6) into

$$\gamma^2+\gamma_1^2+\gamma_2^2=\Gamma^2$$

gives the equation connecting c_0 and Γ

$$s_0^2+s_0'^2+4\frac{c_0c_1}{b}s_1'+\frac{4}{b^2}(c_1^2m_2+2cc_1m_1-c^2c_0^2)=\Gamma^2$$

and the resulting solution has eight independent parameters

$$a,\ a_1,\ b,\ \lambda,\ \lambda_1,\ \Gamma,\ \xi_0,\ \alpha_0$$

Parameter ξ_0 is found from (4.7), while α_0 will appear during the determination of the position of the body in space, based on kinematic equations presented in Sections 1.5 and 1.6 of [13].

BIBLIOGRAPHY

1. Zhukovskii, N.E., O dvizhenii tverdogo tela, imeiushchego polosti napolnennye kapel'noi zhidkost'iu (On the Motion of a Rigid Body with Fluid-Filled Cavities). Collected Works. Vol.3, M.-L., ONTI, 1936,

2. Volterra, V., Sur la théorie des variations des latitudes. Acta Math. Vol.22, 1899.

3. Kowalewski, S.V., Zadacha o dvizhenii tverdogo tela okolo nepodvizhnoi tochki. V Sb. "Dvizhenie tverdogo tela vokrug nepodvizhnoi tochki" (Problem of the Motion of a Rigid Body Around a Fixed Point. In Collected Works, Vol.1, M.-L., Gostekhizdat, 1950.

4. Chaplygin, S.A., Novoe chastnoe reshenie zadachi o vrashchenii tiazhelogo tverdogo tela vokrug nepodvizhnoi tochki. Chaplygin, S.A., Sobr.soch. (New Particular Solution of the Problem of Rotation of a Heavy Rigid Body About a Fixed Point. Collected Works, Vol.1), M.-L., Gostekhizdat, 1948.

5. Kharlamov, P.V., Polinomial'nye resheniia uravnenii dvizheniia tela, imeiushchego nepodvizhnuiu tochku (Polynomial solutions of the equations of motion of a body with a fixed point). *PMM* Vol.29, № 1, 1965.

6. Sretenskii, L.N., O nekotorykh sluchaiakh dvizheniia tiazhelogo tela s giroskopom (On some cases of motion of a heavy rigid body with a gyroscope). Vestn.mosk.gos.Univ., Ser.matem., mekhan,, № 3, 1963.

7. Chaplygin, S.A., Novyi sluchai vrashcheniia tiazhelogo tverdogo tela, podpertogo v odnoi tochke. Chaplygin, S.A., Sobr.soch. (New Case of Rotation of a Heavy Rigid Body, Supported at One Point. Collected works, Vol.1). M.-L., Gostekhizdat, 1948.

8. Kharlamov, P.V., O ravnomernykh vrashcheniiakh tela, imeiushchego nepodvizhnuiu tochku (On uniform rotations of a body with a fixed point). *PMM* Vol.29, № 2, 1965.

9. Kharlamov, P.V., Odno reshenie zadachi o dvizhenii tela, imeiushchego nepodvizhnuiu tochku (A solution for the motion of a body with a fixed point). *PMM* Vol.28, № 1, 1964.

10. Hess, W., Über die Euler'schen Bewegungsgleichungen und über eine neue partikuläre Lösung des Problems der Bewegung eines starren schweren Körpers um einen festen Punkt. Math.Annaln., B.37, H.2, 1890.

11. Grioli, G., Esistenza e determinazione della precessioni regolari dimamicamente possibili per un solido pesante assimetrico. Annali Mat. pura appl., Vol.24, № 3-4, 1947.

12. Kharlamova, E.I., Ob odnom chastnom reshenii uravnenii Eilera-Puassona (On a particular solution of the Euler-Poisson equations). *PMM* Vol.22, № 4, 1959.

13. Kharlamov, P.V., Lektsii po dinamike tverdogo tela, ch.1 (Lectures on Rigid Body Dynamics, Part 1). Izd.Novosibirsk.Univ. 1965.

14. Zhukovskii, N.E., Loksodromicheskii maiatnik Gessa (Hess' Loxodromic Pendulum). Collected Works, Vol.1, M.-L., Gostekhizdat, 1950.

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